about the x-axis. Therefore, from Formula 7, we get

$$S = \int_0^\pi 2\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2}\, dt$$

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi \int_0^\pi r \sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2$$

10.2 EXERCISES

1–2 Find dy/dx.

1. $x = t \sin t, \quad y = t^2 + t$

2. $x = 1/t, \quad y = \sqrt{t}\, e^{-t}$

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = t^4 + 1, \quad y = t^3 + t; \quad t = -1$

4. $x = t - t^{-1}, \quad y = 1 + t^2; \quad t = 1$

5. $x = e^{\sqrt{t}}, \quad y = t - \ln t^2; \quad t = 1$

6. $x = \cos\theta + \sin 2\theta, \quad y = \sin\theta + \cos 2\theta; \quad \theta = 0$

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. $x = 1 + \ln t, \quad y = t^2 + 2; \quad (1, 3)$

8. $x = \tan\theta, \quad y = \sec\theta; \quad (1, \sqrt{2})$

9–10 Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).

9. $x = 6 \sin t, \quad y = t^2 + t; \quad (0, 0)$

10. $x = \cos t + \cos 2t, \quad y = \sin t + \sin 2t; \quad (-1, 1)$

11–16 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

11. $x = 4 + t^2, \quad y = t^2 + t^3$

12. $x = t^3 - 12t, \quad y = t^2 - 1$

13. $x = t - e^t, \quad y = t + e^{-t}$

14. $x = t + \ln t, \quad y = t - \ln t$

15. $x = 2 \sin t, \quad y = 3 \cos t, \quad 0 < t < 2\pi$

16. $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = 10 - t^2, \quad y = t^3 - 12t$

18. $x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$

19. $x = 2\cos\theta, \quad y = \sin 2\theta$

20. $x = \cos 3\theta, \quad y = 2 \sin\theta$

21. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

23–24 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

23. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

24. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

25. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

26. Graph the curve $x = \cos t + 2 \cos 2t$, $y = \sin t + 2 \sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d \sin\theta$, $y = r - d \cos\theta$ in terms of θ. (See Exercise 40 in Section 10.1.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$ in terms of θ. (Astroids are explored in the Laboratory Project on page 629.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

29. At what points on the curve $x = 2t^3$, $y = 1 + 4t - t^2$ does the tangent line have slope 1?

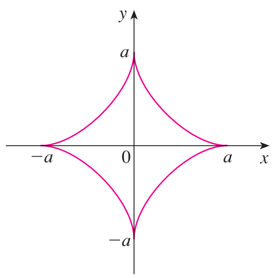
30. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

31. Use the parametric equations of an ellipse, $x = a \cos\theta$, $y = b \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

33. Find the area enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

34. Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Laboratory Project on page 629.)



35. Find the area under one arch of the trochoid of Exercise 40 in Section 10.1 for the case $d < r$.

36. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.

(a) Find the area of $\mathcal{R}$.

(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.

(c) Find the centroid of $\mathcal{R}$.

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t - t^2, \quad y = \frac{4}{3}t^{3/2}, \quad 1 \leqslant t \leqslant 2$

38. $x = 1 + e^t, \quad y = t^2, \quad -3 \leqslant t \leqslant 3$

39. $x = t + \cos t, \quad y = t - \sin t, \quad 0 \leqslant t \leqslant 2\pi$

40. $x = \ln t, \quad y = \sqrt{t + 1}, \quad 1 \leqslant t \leqslant 5$

41–44 Find the exact length of the curve.

41. $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

42. $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \leqslant t \leqslant 3$

43. $x = \dfrac{t}{1 + t}, \quad y = \ln(1 + t), \quad 0 \leqslant t \leqslant 2$

44. $x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

45–47 Graph the curve and find its length.

45. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leqslant t \leqslant \pi$

46. $x = \cos t + \ln\left(\tan \frac{1}{2}t\right), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

47. $x = e^t - t, \quad y = 4e^{t/2}, \quad -8 \leqslant t \leqslant 3$

48. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

49. Use Simpson's Rule with $n = 6$ to estimate the length of the curve $x = t - e^t$, $y = t + e^t$, $-6 \leqslant t \leqslant 6$.

50. In Exercise 43 in Section 10.1 you were asked to derive the parametric equations $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

51–52 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

51. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

52. $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

53. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$$

where e is the eccentricity of the ellipse $\left(e = c/a, \text{ where } c = \sqrt{a^2 - b^2}\right)$.

54. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

CAS **55.** (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?

(b) Use your CAS to find the approximate length of this curve.

CAS **56.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\, du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2)\, du$$

where C and S are the Fresnel functions that were introduced in Chapter 5.

(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?

(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

57–58 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

57. $x = 1 + te^t, \quad y = (t^2 + 1)e^t, \quad 0 \leqslant t \leqslant 1$

58. $x = \sin^2 t, \quad y = \sin 3t, \quad 0 \leqslant t \leqslant \pi/3$

59–61 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

59. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

60. $x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$

61. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

62. Graph the curve

$$x = 2\cos\theta - \cos 2\theta \qquad y = 2\sin\theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

63. If the curve

$$x = t + t^3 \qquad y = t - \frac{1}{t^2} \qquad 1 \leqslant t \leqslant 2$$

is rotated about the x-axis, use your calculator to estimate the area of the resulting surface to three decimal places.

64. If the arc of the curve in Exercise 50 is rotated about the x-axis, estimate the area of the resulting surface using Simpson's Rule with $n = 4$.

65–66 Find the surface area generated by rotating the given curve about the y-axis.

65. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

66. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

67. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

68. Use Formula 2 to derive Formula 7 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

69. The **curvature** at a point P of a curve is defined as

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.

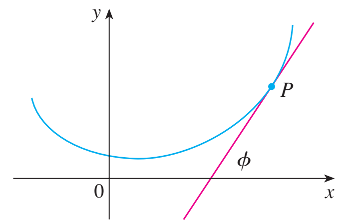
(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$



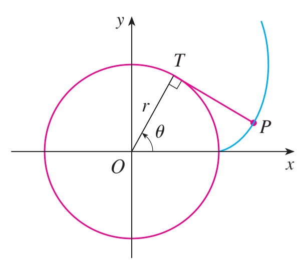
70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

71. Use the formula in Exercise 69(a) to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

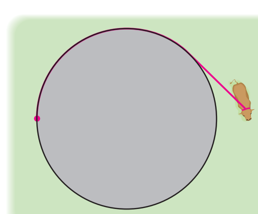
72. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$



74. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.



LABORATORY PROJECT

BÉZIER CURVES

The **Bézier curves** are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910–1999), who worked in the automotive industry. A cubic Bézier curve is determined by four *control points*, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1 - t)^3 + 3x_1 t(1 - t)^2 + 3x_2 t^2(1 - t) + x_3 t^3$$
$$y = y_0(1 - t)^3 + 3y_1 t(1 - t)^2 + 3y_2 t^2(1 - t) + y_3 t^3$$

where $0 \leqslant t \leqslant 1$. Notice that when $t = 0$ we have $(x, y) = (x_0, y_0)$ and when $t = 1$ we have $(x, y) = (x_3, y_3)$, so the curve starts at P_0 and ends at P_3.

1. Graph the Bézier curve with control points $P_0(4, 1)$, $P_1(28, 48)$, $P_2(50, 42)$, and $P_3(40, 5)$. Then, on the same screen, graph the line segments P_0P_1, P_1P_2, and P_2P_3. (Exercise 31 in Section 10.1 shows how to do this.) Notice that the middle control points P_1 and P_2 don't lie on the curve; the curve starts at P_0, heads toward P_1 and P_2 without reaching them, and ends at P_3.

2. From the graph in Problem 1, it appears that the tangent at P_0 passes through P_1 and the tangent at P_3 passes through P_2. Prove it.

3. Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.

4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that gives a reasonable representation of the letter C.

5. More complicated shapes can be represented by piecing together two or more Bézier curves. Suppose the first Bézier curve has control points P_0, P_1, P_2, P_3 and the second one has control points P_3, P_4, P_5, P_6. If we want these two pieces to join together smoothly, then the tangents at P_3 should match and so the points P_2, P_3, and P_4 all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.

10.3 POLAR COORDINATES

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

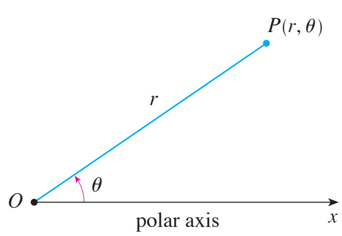


FIGURE 1

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.